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Combined free and forced convection effects on the non-Newtonian flow through a channel

SHANKAR PRASAD MISHRA AND GOURANGA CHARAN DASH

Post-Graduate Department of Mathematics, Utkal Universty, Bhubaneswar-4 Orissa, India.

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The combined effect of free and forced convection on the flow of non-Newtonian liquid between two horizontal parallel walls with a linear axial temperature variation has been studied in this paper. The equations of motion and energy have been solved by two different methods. (i) An approximate method known as perturbation method in which the non-Newtonian parameter is the perturbation parameter, and (ii) Runge-Kutta numerical integration method. The results in both these methods have been compared. The flow phenomenon is characterized by the non-dimensional numbers like R (non-Newtonian number), G (Grashof number), K (Brinkman number) and N (a wall temperature parameter), and the effects of these numbers on the velocity and temperature fields, shear stress and the rate of heat transfer at the walls have been studied.

1. INTRODUCTION

Investigation of natural convection effects in forced horizontal flows are of great physical interest. Similar type of problems have been studied by Sparrow *et al* (1959), Gill & Casal (1962), Gupta (1969), Iqbal *et al* (1969), Hallman (1956) and Ostrach (1954). In this paper our aim is to study the free and forced convection effects on a liquid whose viscosity coefficient is a function of the flow invariants. The equations of motion continuity and heat energy in two-dimensional steady fluid motion are, respectively,

$$\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{\partial p}{\partial x}+\rho F_x+2\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}, \quad \dots \quad (1.1)$$

$$\rho\left[u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right]=-\frac{\partial p}{\partial y}+\rho F_y+\frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}+2\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right), \quad \dots \quad (1.2)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad \dots \quad (1.3)$$

and

$$\rho c\left[u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right]=k\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\phi', \quad \dots \quad (1.4)$$

where $u = u(x, y)$, and $v = v(x, y)$,

are the x and y components of velocity, T is the temperature, ρ is the density of the medium, p is the pressure, c and k are the specific heat capacity and thermal conductivity, respectively, and ϕ' is the dissipation function given by

$$\phi' = \mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2.\right. \quad \ldots \quad (1.5)$$

F_x and F_y are the body forces along x and y axes on unit mass of the fluid.

2. Formulation of Problem

We take x and y axes along and transverse to the parallel plates with origin on the lower plate. At a large distance from the entry, the flow will be fully developed. So we can take all physical variables to depend on y only. The velocity field consistent with the continuity condition (1 3) can be taken in the form

$$u = u(y), \qquad v = 0. \quad \ldots \quad (2.1)$$

Since the plates are horizontal and the x axis is parallel to the plates, in this problem

$$F_x = 0 \text{ and } F_y = -g \quad \ldots \quad (2.2)$$

where g is the acceleration due to gravity. With the help of (2.1) and (2 2), the equations (1.1), (1.2) and (1.4) reduce to

$$0 = -\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left[\mu\frac{\partial u}{\partial y}\right] \quad \ldots \quad (2.3)$$

$$0 = -\frac{\partial p}{\partial y}-\rho g \quad \ldots \quad (2.4)$$

and
$$\rho c u\frac{\partial T}{\partial x} = k\frac{\partial^2 T}{\partial y^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2$$

By virtue of (2.1), the flow invariants are

$$I_1 = 0,\ I_2 = -\frac{1}{4}\left(\frac{du}{dy}\right)^2,\quad I_3 = 0. \quad \ldots \quad (2.6)$$

Hence, it follows from (2.6) that the coefficient of viscosity μ is in general a function of du/dy. We confine our attention following Jones (1961), to the particular class of fluids characterized by

$$\mu = \mu_0\left[1-\alpha\frac{du}{dy}\right] \quad (2.7)$$

where μ_0 and α are physical constants. Since the viscosity coefficient μ is of necessity positive, we must have

$$\mu_0 \geqslant 0 \text{ and } 1-\alpha \frac{\mathrm{d}u}{\mathrm{d}y} \geqslant 0.$$

From (2.3) and (2.7), we get

$$0 = -\frac{\partial p}{\partial x}+\mu_0 \frac{\mathrm{d}^2u}{\mathrm{d}y^2} -\alpha\mu_0 \frac{\mathrm{d}}{\mathrm{d}y} \left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 \qquad \dots \quad (2\ 8)$$

The equation of state is assumed to be

$$\rho = \rho_0[1-\beta(T-T_0)] \qquad \dots \quad (2\ 9)$$

where ρ_0 and T_0 denote the density and temperature of a reference state and β is the coefficient of volume expansion.

From (2.4) and (2.9) we get

$$-\frac{1}{\rho_0} \frac{\partial p}{\partial y} = g[1-\beta(T-T_0)] \qquad \dots \quad (2\ 10)$$

Equation (2 8) can be written as

$$0 = -\frac{1}{\rho_0} \frac{\partial p}{\partial x}+\nu_0 \frac{\mathrm{d}^2u}{\mathrm{d}y^2}-\alpha\nu_0 \frac{\mathrm{d}}{\mathrm{d}y} \left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 \qquad \dots \quad (2.11)$$

where $$\nu_0 = \frac{\mu_0}{\rho_0}.$$

Differentiating (2.10) with respect to x, we get

$$\frac{1}{\rho_0} \frac{\partial^2 p}{\partial x \partial y} = g\beta \frac{\partial}{\partial x} (T-T_0) \qquad \dots \quad (2.12)$$

and differentiating (2.11) with regard to y, we get

$$0 = -\frac{1}{\rho_0} \frac{\partial^2 p}{\partial x \partial y}+\nu_0 \frac{\mathrm{d}^3u}{\mathrm{d}y^3} -\alpha\nu_0 \frac{\mathrm{d}^2}{\mathrm{d}y^2} \left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2. \qquad \dots \quad (2.13)$$

From (2.12) and (2.13) we get,

$$\nu_0 \frac{\mathrm{d}^3u}{\mathrm{d}y^3} -\alpha\nu_0 \frac{\mathrm{d}^2}{\mathrm{d}y^2} \left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 = g\beta \frac{\partial}{\partial x} (T-T_0). \qquad \dots \quad (2.14)$$

Let us now introduce the non-dimensional parameters

$$u = U_0 w \qquad y = h\eta, \quad x = P_e h\xi \qquad \dots \quad (2.15)$$

where U_0, h and P_e represent the bulk mean velocity, distance between the plates and Peclet number, respectively. With these parameters, (2.14) takes the form

$$\frac{\mathrm{d}^3w}{\mathrm{d}\eta^3} -R \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} \left(\frac{\mathrm{d}w}{\mathrm{d}\eta}\right)^2 = \frac{g\beta h^3}{\nu_0 U_0 P_e} \frac{\partial}{\partial \xi} (T-T_0), \qquad \dots \quad (2.16)$$

where $R = (\alpha U_0/h)$, the non-Newtonian parameter. The boundary conditions on u are

$$u = 0 \text{ at } y = 0 \text{ and } y = h, \int_0^h u\mathrm{d}y = U_0 h. \qquad \dots (2.17)$$

which now become

$$w = 0 \text{ at } \eta = 0 \text{ and } \eta = 1, \int_0^1 w\mathrm{d}\eta = 1 \qquad \dots (2.18)$$

Since w is a function of η only, (2.16) is valid only if the right hand side is independent of ξ. This is clearly satisfied if the temperature distribution is given by

$$T - T_0 = A_1\xi + Y(\eta) \qquad \dots (2.19)$$

where A_1 is a constant and $Y(\eta)$ is function of η only. Equation (2.19) corresponds to uniform axial temperature variation along the channel walls. Now from (2.16) and (2.19) we have

$$\frac{\mathrm{d}^3 w}{\mathrm{d}\eta^3} - R\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\left(\frac{\mathrm{d}w}{\mathrm{d}\eta}\right)^2 = G. \qquad \dots (2.20)$$

where
$$G = \frac{g\beta h^2 A_2}{\nu_0 U_0}, \quad A_2 = \frac{A_1}{P_e}, \quad P_e = \frac{U_0 h}{\lambda} \quad \lambda = \frac{k}{\rho c} \qquad \dots (2.21)$$

The energy equation (2.5), with the help of (2.7) gives

$$u\frac{\partial(T-T_0)}{\partial x} = \lambda\frac{\partial^2}{\partial y^2}(T-T_0) + \frac{\nu_0}{c}\left[\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 - \alpha\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3\right] \qquad \dots (2.22)$$

With the transformations (2.15), the equation (2.22) takes the form

$$w = \frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2} + K\left[1 - R\frac{\mathrm{d}w}{\mathrm{d}\eta}\right]\left[\frac{\mathrm{d}w}{\mathrm{d}\eta}\right]^2 \qquad \dots (2.23)$$

where
$$Y(\eta) = A_2 P_e \phi; \quad K = \frac{\nu_0 U_0^2}{c P_e A_2 \lambda} \qquad \dots (2.24)$$

The boundary conditions are given in (2.18). As for the temperature boundary conditions, we take the reference temperature T_0 such that the temperature of the lower wall ($\eta = 0$) is $T_0 + A_1\xi$ and this, by virtue of (2.19) implies that $Y(0) = 0$. Hence, the boundary conditions on $\phi(\eta)$ satisfying (2.23) are

$$\phi(0) = 0 \quad \text{and} \quad \phi(1) = \frac{Y(1)}{P_e A_2} = N. \qquad \dots (2.25)$$

Since w depends on G and R, it follows from (2.23) and (2.25) that $\phi(\eta)$ and hence the temperature distribution depends on the parameters G, R, K and N, where

K is known as the Brinkman number and N is the wall temperature parameter. We first solve (2.20) and the expression for w is then substituted in (2.23) to get an expression for ϕ.

3. Solution of Equations

Integrating (2.20) twice with respect to η we get

$$\frac{dw}{d\eta}-R\left(\frac{dw}{d\eta}\right)^2=\tfrac{1}{2}G\eta^2+\gamma\eta+\delta \quad \ldots \quad (3.1)$$

where γ and δ are the constants of integration.

An exact solution of (3.1) can be obtained. But the expression for w will be very complicated and the energy equation (2.23) will be so very complicated that the physical aspects of the problem will be completely marred. Hence we shall obtain solutions in two different methods.

Perturbation Solution

We take the non-Newtonian parameter R as the perturbation parameter and assume it to be so small that its cubes and higher powers can be neglected. We then write

$$w=\sum_{n=0}^{\infty} w_n R^n \quad \ldots \quad (3.2)$$

$$\gamma=\sum_{n=0}^{\infty} \gamma_n R^n \quad \ldots \quad (3.3)$$

$$\delta=\sum_{n=0}^{\infty} \delta_n R^n \quad \ldots \quad (3.4)$$

Substituting (3.2)—(3.4) into (3.1), comparing the coefficients of different powers of R and solving with the boundary conditions on w_0, w_1, w_2, which are from (2.18)

$$w_0=w_1=w_2=0 \text{ when } \eta=0,\ \eta=1, \quad . \quad (3.5)$$

$$\int_0^1 w_0 d\eta=1, \qquad \int_0^1 w_1 d\eta=0, \qquad \int_0^1 w_2 d\eta=0, \quad . \quad (3.6)$$

We get the zeroth order solution :

$$w^{(0)}=\frac{1}{6}G\eta^3+\tfrac{1}{2}\gamma_0\eta^2+\delta^0\eta; \quad . \quad (3.7)$$

First order solution :

$$w^{(1)}=\frac{1}{6}G\eta^3+\tfrac{1}{2}\eta^2\gamma_0+\delta_0\eta+R\frac{1}{20}G^2\eta^5+\frac{1}{4}G\gamma_0\eta^4+\frac{1}{3}(\gamma_0+G\delta_0)\eta^3+$$

$$(\gamma_0\delta_0+\tfrac{1}{2}\gamma_1)\eta^2+(\delta_0^2+\delta_1)\eta; \quad . \quad (3.8)$$

Second order solution :

$$w^{(2)}=\frac{1}{6}G\eta^3+\tfrac{1}{2}\gamma_0\eta^2+\delta_0\eta+R\Big[\frac{1}{20}G^2\eta^5+\frac{1}{4}G\gamma_0\eta^4+\frac{1}{3}(\gamma_0^2+G\delta_0)\eta^3+$$

$$(\gamma_0\delta_0+\tfrac{1}{2}\gamma_1)\eta^2+(\delta_0^2+\delta_1)\eta\Big]+R^2\Big[\frac{1}{28}G^3\eta^7+\frac{1}{4}G^2\gamma_0\eta^6+\frac{3}{5}\gamma_0^2G+$$

$$\frac{3}{10}G^2\delta_0)\eta^5+\Big(\frac{3}{2}\gamma_0\delta_0G+\tfrac{1}{2}\gamma_0^3+\frac{1}{4}G\gamma_1\Big)\eta^4+\Big(\frac{1}{3}G\delta_1+2\gamma_0^2\delta_0+G\delta_0^2+\frac{2}{3}\gamma_0\gamma_1\Big)\eta^3+$$

$$(3\delta_0^2\gamma_0+\gamma_0\delta_1+\gamma_1\delta_0+\tfrac{1}{2}\gamma_2)\eta^2+(2\delta_0^3+2\delta_0\delta_1+\delta_2)\eta\Big], \quad \ldots \quad (3.9)$$

where

$$\gamma_0=-\tfrac{1}{2}(G+24),\quad \delta_0=\frac{1}{12}(G+72),\quad \gamma_1=\frac{4}{5}G,\quad \delta_1=-\Big[12+\frac{2}{5}G+\frac{1}{720}G^2\Big],$$

$$\gamma_2=-\Big[\frac{9}{56}G^3+\frac{15}{14}G^2\gamma_0+\frac{12}{5}\gamma_0^2G+\frac{6}{5}G^2\delta_0+\frac{27}{5}\gamma_0\delta_0G+\frac{9}{5}\gamma_0^3+\frac{9}{10}G\gamma_1+G\delta_1$$

$$+6\gamma_0^2\delta_0+3G\delta_0^2+2\gamma_0\gamma_1+6\delta_0^2\gamma_0+2\gamma_0\delta_1+2\gamma_1\delta_0\Big],$$

$$\delta_2=\frac{5}{112}G^3+\frac{2}{7}G^2\gamma_0+\frac{3}{5}\gamma_0^2G+\frac{3}{10}G^2\delta_0+\frac{6}{5}\gamma_0\delta_0G+\frac{2}{5}\gamma_0+\frac{1}{5}G\gamma_1+$$

$$\frac{1}{6}G\delta_1+\gamma_0^2\delta_0+\tfrac{1}{2}G\delta_0^2+\frac{1}{3}\gamma_0\gamma_1-2\delta_0^3-2\delta_0\delta_1.$$

If we restrict our discussions to liquids which are very slightly non-Newtonian in character we can take (3.9) as the approximate solution for velocity distribution The constant G appearing in the solution (3.9) is a quantity characterizing the extent of influence exerted by the bouyancy forces on the forced convection flow. To avoid the algebraical complications we substitute the first order solution from (3.8) in (2.23) and integrating twice with respect to η and using the conditions (2.25), we get

$$\phi=\frac{1}{120}G\eta^5+\frac{1}{24}\gamma_0\eta^4+\frac{1}{6}\delta_0\eta^3+R\Big[\frac{1}{840}G^2\eta^7+\frac{1}{120}G\gamma_0\eta^6$$

$$+\frac{1}{60}(\gamma_0^2+G\delta_0)\eta^5+\frac{1}{24}(\gamma_1+2\gamma_0\delta_0)\eta^4+\frac{1}{6}(\delta_0^2+\delta_1)\eta^3\Big]$$

$$-K\Big[\frac{1}{120}G^2\eta^6+\frac{1}{20}G\gamma_0\eta^5+\frac{1}{12}(\gamma_0^2+\delta_0G)\eta^4+\frac{1}{3}\gamma_0\delta_0\eta^3+\tfrac{1}{2}\delta_0^2\eta^2$$

$$+R\left\{\frac{1}{448}G^3\eta^8+\frac{1}{56}G^2\gamma_0\eta^7+\frac{1}{20}(\tfrac{1}{2}G^2\delta_0+G\gamma_0^2)\eta^6+\frac{1}{20}(3G\delta_0\gamma_0+\gamma_0^3+\gamma_1 G)\eta^5\right.$$

$$+\frac{1}{12}(3\gamma_0^2\delta_0+\frac{3}{2}G\delta_0^2+2\gamma_1\gamma_0+G\delta_1)\eta^4+\frac{1}{6}(3\gamma_0\delta_0^2+2\gamma_1\delta_0+2\gamma_0\delta_1)\eta^3$$

$$\left.+\tfrac{1}{2}(\delta_0^3+2\delta_0\delta_1)\eta^2\right\}\Bigg]+C_1\eta \tag{3.10}$$

where

$$C_1 = N-\Bigg[\frac{G}{720}+\tfrac{1}{2}-K\left(6+\frac{G}{15}+\frac{G^2}{1440}\right)+R\left\{\frac{G^2}{30240}+\right.$$

$$\left.\left.+K\left(\frac{12}{5}+\frac{G}{5}+\frac{17G^2}{25200}-\frac{G^3}{60480}\right)\right\}\right]$$

Numerical Integration by Runge-Kutta Method

For solving (2.20) and (2 23) we adopt Runge-Kutta-Gill technique and proceed with the step by step integration as illustrated by Ralston & Wilf (1960). The following substitutions transform equations (2.20) and (2 23) into six first order linear equations for numerical integration. :

$$y_1 = \int_0^\eta w\,d\eta, \quad y_2 = \frac{dw}{d\eta}, \quad y_3 = w$$

$$y_4 = \frac{d^2w}{d\eta^2}, \quad y_5 = \phi, \quad y_6 = \frac{d\phi}{d\eta} \tag{3.11}$$

With the help of (3.11) we form the required set of primary equations

$$y_1' = y_3, \quad y_2' = y_4, \quad y_3' = y_2, \quad y_4' = \frac{G+2Ry_4^2}{1-2Ry_3}, \quad y_5' = y_6,$$

$$y_6' = y_3-K(1-Ry_2)y_2^2. \tag{3.12}$$

The boundary conditions given by (2.18) and (2.25) take the form

$$y_1(0) = y_3(0) = y_5(0) = 0, \quad y_1(1) = 1, \quad y_3(1) = 0, \quad y_5(1) = N \tag{3.13}$$

Denoting by e any one of the unknown values $y_2(0)$, $y_4(0)$ and $y_6(0)$, and differentiating the primary equations (3.12) with respect to e and letting $\frac{dy_i}{de} = p_i$ $(i = 1, 2 \ldots 6)$ the set of auxilary equation is

$$p_1' = p_3, \quad p_2' = p_4, \quad p_3' = p_2,$$

$$p_4' = \frac{2R[2y_4p_4-4Ry_3y_4p_4+Gp_3+2Ry_4^2p_3],}{(1-2Ry_3)^2} \tag{3.14}$$

$$p_5' = p_6, \quad p_6' = p_3-2Ky_2p_2+3RKy_2^2p_2.$$

The boundary conditions for the set of auxiliary equations are

$$p_i = \delta_{2i}, \quad p_i = \delta_{4i} \quad p_i = \delta_{6i}$$
$$(i = 1, 2, \ldots, 6)) \qquad \ldots \quad (3.15)$$

and δ_{ij} is the Kronecker's delta.

The step by step integrations of (3.12) subject to the boundary conditions (3.13) are performed in (0, 1) with a step length 0.05. The rough starting values of $y_2(0)$, $y_4(0)$, $y_6(0)$ supplied to start the integration were corrected by a self-iterative corrective procedure (Fox 1962) and this was done with the aid of the auxiliary equations (3.14) subject to boundary conditions (3.15).

The numerical integrations were performed on an IBM1130 digital electronic computer (Utkal University, India).

4. Conclusions

Velocity distribution

Figure 1 represents the velocity distribution for different values of G and R. It is seen from this figure that the velocity distribution becomes asymmetric in the presence of buoyancy force ($G \neq 0$). In the lower half of the channel, the

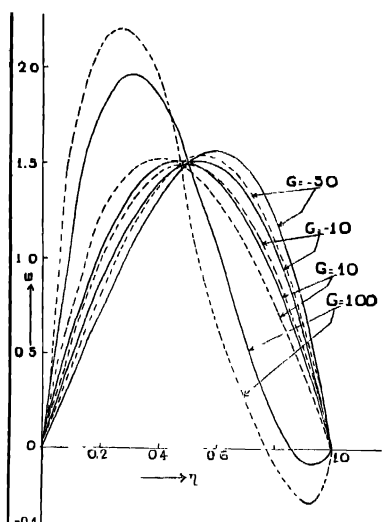

Figure 1. Velocity distribution for different values of G
$R = 0$,——— ; $R = 0.05$, - - - -

increase in the value of G increases the velocity at any point and an opposite effect is observed in the other half of the channel. It is interesting to note that for higher values of G, say $G = 100$, in the vicinity of the upper plate, the velocity becomes negative. For a viscous liquid with constant viscosity coefficient, the velocity curves for different values of G intersect at $\eta = 0.5$, that is, at the middle of the channel. The non-Newtonian character of the liquid increases the velocity at any point in the lower half of the channel up to $0 < \eta < 0.473$. In the other half, that is $0.473 < \eta < 1$, the non-Newtonian parameter decreases the velocity at any point in the liquid. It is also clear that the non-Newtonian nature of the liquid shifts the point of intersection of the velocity curves for different values of G to a point nearer to the lower plate.

Table 1 represents the values of velocity at some points by both the methods. This table shows that the values agree well by both the methods, which shows that the perturbation solution is effective in this problem.

Table 1. Values of velocity at some points. ($G = -10$)

	Perturbation method		Runge-Kutta method	
$R =$ / $\eta =$	0.00	0.10	0.00	0.10
0.2	0.8800	1.10267	0.8799985	1.1677565
0.4	1.4000	1.54370	1.3999950	1.4626460
0.6	1.4800	1.33382	1.4799921	1.3097522
0.8	1.0400	0.7982	1.0399999	0.8106164

Shearing stress at the walls

The shearing stresses at the walls are given by

$$\tau_0 = p_{xy}\frac{h}{\mu_0 U_0}\Big]_{\eta=0} = \left(\frac{\mathrm{d}w}{\mathrm{d}\eta}\right)_{\eta=0} - R\left(\frac{\mathrm{d}w}{\mathrm{d}\eta}\right)^2_{\eta=0}$$

$$\tau_1 = p_{xy}\frac{h}{\mu_0 U_0}\Big]_{\eta=1} = \left(\frac{\mathrm{d}w}{\mathrm{d}\eta}\right)_{\eta=1} - R\left(\frac{\mathrm{d}w}{\mathrm{d}\eta}\right)^2_{\eta=1}$$

Table 2 represents the computed values of the shearing stress at the walls. At both the walls, for a viscous liquid, the shearing stress increases with G and the non-Newtonian character of the fluid increases it further at both the walls. Thus there is no flow separation in this case. But more and more cooling at the lower plate (which corresponds to negative values of G mentioned before) causes a progressive decrease in the values of the skin-friction. This leads us to

Table 2. Skin friction at the plates.

$\eta =$	$G =$ −20		10			$R =$
0			6 8333		14.3333	0.00
	4 6764		7.5791	8 6532	20.6261	0 02
1	−7.0666	−6.8333	−5 1666	−4 3332	2.3334	0.00
	−7 0926	−6 3312	−4.7179	−3.8602	4.7867	0.02

the conclusion that for very high negative values of G, the velocity becomes unstable.

Temperature Field

Figure 2 and table 3 represent the temperature field for different values of G and R when the wall temperature parameter $N = 1$. An examination of the

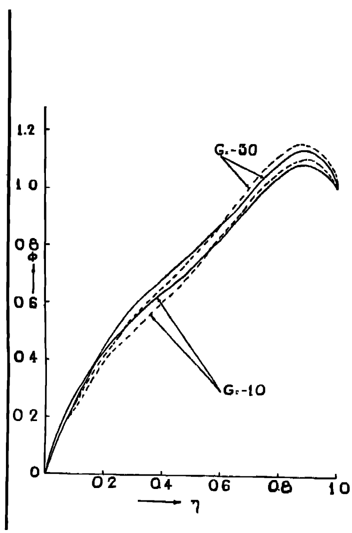

Figure 2. Temperature distribution for different values of G; $N = 1$, $K = 0.5$, $R = 0$, ——— ; $R = 0.05$, - - - -

figure shows that the temperature at any point in a liquid layer near the plate $\eta = 0$ (lower plate) decreases as being negative G increases. But beyond this liquid layer an opposite effect is observed. The non-Newtonian character of the liquid decreases the temperature in the liquid layer near the lower plate. When $G = -10$, the temperature curves for Newtonian and non-Newtonian cases intersect at about $\eta = 0.5533$. When $G = -30$, the point of intersection occurs at $\eta = 0.60$. This shows that being negative as G increases, the points of inter-

Table 3. Temperature distribution for positive values of G; $K = 0.5$; $N = 1\ 0$

$\eta =$ \ $G =$	10	20	100	$R =$
0.2	0.4442	0.4623	0 8741	0 00
	0 4306	0 4485	1.0340	0 02
0.4	0 6292	0.6451	1.2106	0.00
	0 6214	0 6407	1 5669	0.02
0 5	0.7252	0 7447	1 3697	0 00
	0 7252	0.7493	1 7722	0 02
0.6	0 8341	0 8548	1.4591	0 00
	0.8403	0 8651	1 8412	0 02
0 8	1.0303	1 0344	1.3343	0.00
	0.9992	1 0399	1.5387	0.02

section of the curves for Newtonian and non-Newtonian cases shift towards the upper plate. Table 3 shows the temperature distribution for positive values of G. An examination of this table shows that the temperature at any point increases with G. For small values of G, for $0 < G \leqslant 20$ (nearly), the non-Newtonian parameter first decreases the temperature in the liquid layer $0 \leqslant \eta \leqslant 0.5$ and then an opposite effect is observed. For very high values of G say ($G = 100$), the non-Newtonian parameter increases the temperature at every point in the channel.

Rate of heat transfer from the walls

The Nusselt number at both the walls is given by $(\mathrm{d}\phi/\mathrm{d}\eta)_{\eta=0}$ and $(\mathrm{d}\phi/\mathrm{d}\eta)_{\eta=1}$ and increases with the increase in the value of the Grashof number G and the non-Newtonian parameter R decreases it. Table 4 yields this conclusion. In this table the upper line in each row represents the values of the Nusselt number obtained by the Runge-Kutta method and in the lower line by the perturbation method. The results, almost in every case agree well.

Table 4. Nusselt Number. $K = 0.5$ and $N = 1$

$\eta =$ \ $G =$	-20	-10	10	20	$R =$
0	3.0000	3.2152	3.8541	4.2777	0.00
	2 9999	3.2152	3 8542	4.2777	
	2.9740	3.1773	3.8040	4.2406	0.02
	2.9527	3.1507	3.5914	4.0010	
1	−2 2777	−1.8541	−1.2152	−0.9995	0.00
	−2.4835	−2.1103	−1.2820	−0.8732	
	−2.3452	−1 8882	−1.2851	−1.1493	0.02
	−2.5278	−2.1283	−1.2828	−0.8927	

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